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Interface tunnel cracks in a composite anisotropic space☆

A.F. Krivoi, G.Ya. Popov

Odessa, Ukraine

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ABSTRACT

Exact solutions of the problem of tunnel cracks in the plane between two anisotropic half-spaces which are in conditions of generalized plane deformation (without the presence of planes of elastic symmetry) are obtained. Using the proposed procedure, which rests on constructed solutions of the Riemann matrix problem in the space of generalized functions of slow growth, the problem is reduced to a system of singular integral equations. Exact solutions of this system are constructed, which enable the conditions for which zones of overlap of the crack surfaces to be obtained, as well as formulae for calculating the dimensions of these zones, and enable the normal fracture stresses and limit values of the stress intensity factors to be determined. The behaviour of these quantities for different combinations of materials of the monoclinic and orthorhombic systems for orthogonal transformations of the principal axes of symmetry is investigated.

The generalized method of integral transformations was used in Ref. 1 to solve problems of interface defects in an inhomogeneous isotropic plane. The method was then generalized in Ref. 2 to the case of a composite anisotropic plane. In both cases the problems were considered in classes of piecewise-differentiable functions, which imposed corresponding limitations on the loading and complicated the basis of the constructions.

Below, in order to eliminate these drawbacks, the problem of interface tunnel cracks in a composite anisotropic space under conditions of a two-dimensional stress-strain state, is formulated in the form of a boundary-value problem for a system of differential equations in the components of the stress tensor and the displacement vector in the space of generalized functions of slow growth $S'(R^2)$ and is reduced to a Riemann matrix problem in the space $S'(R^2)$. This approach enables loading of the most general form to be considered both on the crack and in the medium, and enables one to determine the number of arbitrary functions which appear in the solutions by virtue of the presence of lines and points of discontinuity.

Note that crack problems have been considered in the plane formulation by many researchers using different methods (for example Refs. 3–7), but only qualitative results are known in the most general formulation for an anisotropic space.[8,9]

1. Formulation of the problem and reduction of the problem to a Riemann matrix problem in $S'(R^2)$

Suppose a space, consisting of two dissimilar anisotropic half-spaces, incompletely coupled in the $x = 0$ plane, is in a two-dimensional stress-strain state (without planes of elastic symmetry[10]). In the $x = 0$ plane there are through cracks, occupying r strips

$$\Pi_j = \{(y, z) | y \in l_j = (a_j, b_j), z \in (-\infty, \infty)\}, \quad a_1 < b_1 < \ldots < a_r < b_r, \quad j = 1, 2, \ldots, r$$

It is assumed that the following functions are known on the surfaces of the cracks

$$\zeta_k^{\pm}(y) = \psi_k^{\pm}(y), \quad k = 1, 2, 3; \quad y \in l_0 = \bigcup_{j=0}^{r} l_j \tag{1.1}$$

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E-mail address: krivoi-odessa@ukr.net (A.F. Krivoi).

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where

$$\{\zeta_k^\pm\}^6 = \{\upsilon_1, \upsilon_3, \upsilon_4, \partial\upsilon_6/\partial y, \partial\upsilon_7/\partial y, \partial\upsilon_8/\partial y\}|_{x=\pm 0},$$
$$\boldsymbol{\upsilon} = \{\upsilon_k(x, y)\}_{k=1,2,\dots,8} = \{\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}, u, \upsilon, w\} \tag{1.2}$$

and $\psi_k^\pm(y)(k = 1, 2, 3)$ is an arbitrary load, applied to the surfaces of the cracks, which ensures generalized plane deformation.[10] We will assume that the stresses vanish at infinity, since otherwise, in view of the linearity, the problem cannot be reduced[1] to the formulation considered if a solution of a corresponding problem without a crack is constructed.

The solution of the problem will be obtained if we find the remaining functions (1.2). For this purpose it is necessary to establish integral relations in the $x = 0$ plane which connect the differences (jumps) and sums of the components of the displacement vector and the stress tensor

$$\boldsymbol{\chi}^\pm = \{\chi_k^\pm(y)\}^6, \quad \chi_k^\pm(y) = \zeta_k^+(t) \pm \zeta_k^-(y), \quad y \in \mathrm{R} \tag{1.3}$$

For this we can obtain a boundary-value problem for the vector υ, starting from the equations of equilibrium and the generalized Hooke's law in the subspace $S_1'(R^2)$ (i.e., in the subspace of generalized functions of slow growth $g(x, y) \in S'(R^2)$, for which $c_x(g) \le 1$, where $c_x(g)$ is the order of the singularity[11] with respect to the variable x). This is a problem in Fourier transformants equivalent to the following matrix equation

$$\mathbf{M}_*^+\mathbf{V}^+(\alpha, \beta) - \mathbf{M}_*^-\mathbf{V}^-(\alpha, \beta) = \mathbf{f}_0, \quad \mathbf{V}^\pm = \{V_k^\pm\}^8 \tag{1.4}$$

where

$$\mathbf{M}_*^\pm = \pm\begin{Vmatrix} \mathbf{D} & \mathbf{O}_{3\times 3} \\ \mathbf{B}^\pm & \mathbf{D}^T \end{Vmatrix}, \quad \mathbf{D} = -i\begin{Vmatrix} \alpha & 0 & \beta & 0 & 0 \\ 0 & \beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha & \beta \end{Vmatrix}, \quad \mathbf{B}^\pm = \{\beta_{kj}^\pm\}^5$$

$$V_k^\pm(\alpha, \beta) = F[\upsilon_k^\pm] \in S_\pm'(\mathrm{R}^2), \quad \mathbf{f}_0 = \{\chi_{*,1}^-, \chi_{*,2}^-, \chi_{*,3}^-, \chi_{*,4}^-, 0, \chi_{*,5}^-, \chi_{*,6}^-, 0\}$$

$$S_\pm'(\mathrm{R}^2) = \{g^\pm \in S'(\mathrm{R}^2) \,|\, \mathrm{supp}\, g_\pm = \mathrm{R}_\pm \times \mathrm{R}\}, \quad \chi_{*,k}^\pm(\beta) = F_2[\chi_k^\pm] \in S'(\mathrm{R})$$

F and F_2 are Fourier transformation operators correspondingly two-dimensional and with respect to the variable y, $\mathbf{O}_{k\times j}$ is a null rectangular $k \times j$ matrix, $\beta_{kj}^\pm$ are the reduced strain coefficients for an anisotropic medium[10] for the upper ($x > 0$) and lower ($x < 0$) half-spaces respectively.

Suppose $H_m(R)$ is a class of functions $f_z(\beta) \in S'(R)$, analytic in the parameter $z = \alpha + i\omega$ at each finite point of the complex plane, with the exception, possibly, of the lines $\mathrm{Im} z = 0$, and, when $|\mathrm{Im} z| > \varepsilon > 0$ and for a certain integer m satisfy the estimate

$$|f_z(\beta)| \le A_\varepsilon(1 + |z|)^m, \quad A_\varepsilon < \infty$$

The function $f_\alpha(\beta) \in S'(R)$ allows of an analytical representation in the variable α, if a function $f_z(\beta) \in H_m(R)$ exists such that (in the sense of convergence in the space $S(R)$)

$$\lim_{\varepsilon\to 0}(f_{\alpha+i\varepsilon}(\beta) - f_{\alpha-i\varepsilon}(\beta)) = f_\alpha^+(\beta) - f_\alpha^-(\beta) = f(\beta) \tag{1.5}$$

Suppose $\Omega_m'(R^2)$ is a space of generalized functions $f \in S'(R^2)$, for which functions yielding analytical representation (1.5) with respect to the variable α, belong to the class $H_m(R)$. Suppose $\Omega_{\pm,m_\pm}'(R^2)$ is a subspace of the functions $f_\pm \in \Omega_{m_\pm}'(R^2)$ for which the functions $f_z^\pm(\beta) \in H_{m_\pm}(R)$, which yield an analytic representation for $\pm\mathrm{Im} z < 0$ respectively, have the form

$$f_z^\pm(\beta) = M_{m_\pm}, \quad M_m(z, \beta) = \sum_{k=0}^m z^k \varphi_k(\beta), \quad (M_m \equiv 0, m < 0)\ \varphi_k \in S'(\mathrm{R}) \tag{1.6}$$

The following assertions hold[1]

Theorem 1.1. *Suppose*

$$g(x, y) \in S_p'(\mathrm{R}^2), \quad g(x, y) = \partial^{p+n_2} g_0(x, y)/\partial x^p \partial y^{n_2}$$

where g_0 is a continuous slow growing function, which allows of the representation

$$g_0 = x^{n_0} g_*(x, y), \quad n_0 \ge 0, \quad g_*(0, y) \ne 0$$

[1] For their proof see Krivoi A. F. A method of solving the Riemann problem with respect to a part of variables in the space of generalized slow growing functions: Odessa: OVIMU; 1988. Deposited at VINITI 24 October 1988, No. 7605-V88.

Then

$$f = F[g] \in \Omega'_{p-n_0-1}(R^2)$$

Theorem 1.2. *Suppose*

$$g_\pm \in S'_{\pm,\,p_\pm}(R^2) = S'_\pm(R^2) \cap S'_{p_\pm}(R^2)$$

Then

$$f_\pm = F[g_\pm] \in \Omega'_{\pm,\,m_\pm}(R^2), \quad m_\pm = p_\pm - 1.$$

Taking these theorems into account, Eq. (1.4) can be regarded as the boundary condition of the Riemann matrix problem in $S'(R^2)$ space with respect to the parameter α, for determining the vector functions

$$V^\pm = \{V_k^\pm\}^8, \quad V_k^\pm \in \Omega'_{\pm,\,0}(R^2)$$

We will solve this problem, basing on the solution of the scalar problem in $S'(R^2)$ space described below.

2. A method of solving the Riemann boundary-value problem with respect to one variable in $S'(R^2)$ space

In the scalar form, the Riemann boundary-value problem with respect to one variable in $S'(R^2)$ space consists of the following: it is required to obtain two functions $f_\pm \in \Omega'_{\pm,m_\pm}(R^2)$ such that

$$(f_+, \varphi) = (f_-, G(\alpha,\beta)\varphi) + (q, \varphi), \quad (q \in S'(R^2), \varphi \in S(R^2)), \quad \beta \in R \tag{2.1}$$

where q is a specified function such that $g = F^{-1}[q] \in S'_n(R^2)$; $G \in \Theta_\mu$, $G \neq 0$, Θ_μ is a class of multipliers in $S(R^2)$, Hölder with respect to the parameter α.

In the spaces considered the jump problem and Liouville's theorem allow of the following generalizations.

Theorem 2.1. *If* $f(\alpha,\beta) \in \Omega'_p(R^2)$, *the following representation holds*

$$f = f_+ - f_-, \quad f_\pm \in \Omega'_{\pm,\,p}(R^2) \tag{2.2}$$

where $f_\pm$ are defined apart from functions of the form $M_p(\alpha,\beta)$ (1.6).

Proof. The following representation holds[12]

$$g = g_+ - g_-, \quad g = F^{-1}[f] \in S'_{p+1}(R^2), \quad g_\pm \in S'_{\pm,\,p+1}(R^2) \tag{2.3}$$

The functions $g_\pm$ are defined, apart from the same function of the form

$$\eta_{p-1}(x,y) = \sum_{k=0}^{p-1} \delta^{(k)}(x)\varphi_k^0(y), \quad \varphi_k^0 \in S'(R); \quad \eta_l \equiv 0,\ l < 0 \tag{2.4}$$

where $\delta(x)$ is the Dirac function. A Fourier transformation of (2.3), taking Theorem 1.1 into account, leads to representation (2.2). The functions $f_\pm$ are defined, apart from a Fourier transformation of function (2.4).

Theorem 2.2. *Suppose*

$$f_\pm(\alpha,\beta) \in \Omega'_{\pm,\,m_\pm}(R^2) \tag{2.5}$$

Then, if

$$(f_+, \varphi) = (f_-, \varphi), \quad \varphi \in S(R^2) \tag{2.6}$$

then

$$f_+ = f_- = M_p(\alpha,\beta), \quad p \le \min\{m_+, m_-\} \tag{2.7}$$

Proof. A Fourier transformation of Eq. (2.6) and Theorem 1.2 lead to the relation

$$(g_+, \varphi(x,y)) = (g_-, \varphi(x,y)); \quad g_\pm = F^{-1}[f_\pm] \in S'_\pm(R^2), \quad \varphi \in S(R^2)$$

This equality is possible if the functions $g_\pm$ are concentrated at the intersection of the carriers of the function from the spaces $S_+'(R^2)$ and $S_-'(R^2)$, i.e. on the line $x = 0$. Consequently, for a certain integer p the representation $g_+ = g_- = \eta_{p-1}$ holds. Its Fourier transformation, taking into account the fact that $M_p = F[\eta_{p-1}]$, leads to the required relation. Since inclusion (2.5) holds, inequality (2.7) holds.

We will now solve problem (2.1). Suppose the index of the coefficient is bounded: $\mathrm{Ind}_\alpha G = k < \infty$. Then the following representation[13] holds

$$G(\alpha,\beta) = \left(\frac{\alpha-i}{\alpha+i}\right)^k \frac{X_+}{X_-}; \quad X_\pm(\alpha,\beta) = \lim_{z\to\alpha\pm i0} X(z,\beta)$$
$$X(z,\beta) = \exp(K_\beta(z)); \quad K_\beta(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \ln\left[\left(\frac{\tau-i}{\tau+i}\right)^{-k} G(\tau,\beta)\right]\frac{d\tau}{\tau-z} \tag{2.8}$$

where $X_\pm \in \Theta_\mu$ are boundary values of functions, bounded at infinity, and analytic with respect to the variable α in the upper and lower half-planes respectively. Representation (2.8), Theorems 1.2 and 2.1, and also the inclusion $(\alpha+i)^k \in \Theta_\mu$ enable us to rewrite condition (2.1) in the form

$$(f_+^0, \varphi) = (f_-^0, \varphi); \quad f_\pm^0 = f_\pm(\alpha,\beta)(\alpha\pm i)^k X_\pm^{-1} - q_k^\pm, \quad \varphi \in S(\mathrm{R}^2)$$
$$q_k = (\alpha+i)^k q X_+^{-1} = q_k^+ - q_k^-, \quad q_k^\pm \in \Omega'_{\pm,\,n+k-1}(\mathrm{R}^2) \tag{2.9}$$

It is obvious that the functions $f_\pm^0$ belong to the subspaces $\Omega'_\pm(R^2)$ respectively.

Suppose $k \geq 0$. Then if $m \geq n-1$ $(m = \min\{m_+, m_-\})$ we have $f_\pm^0 \in \Omega'_{\pm,m+k}(R^2)$, and consequently, on the basis of Theorem 2.2 $f_\pm^0 = M_{m+k}(\alpha,\beta)$, where M_{m+k} are functions of the form (1.6). The solution of problem (2.1) in this case takes the form

$$f_\pm(\alpha,\beta) = (\alpha\pm i)^{-k} X_\pm(M_{m+k} + q_k^\pm) \in \Omega'_{\pm,\,m}(\mathrm{R}^2) \tag{2.10}$$

If $m < n-1$, then, according to Theorems 1.1 and 2.1, for a solution of problem (2.1) to exist in the subspaces $\Omega'_{\pm,m}(R^2)$ it is necessary that, in the relation $g_k = \partial^{k+n+n_2} g_0 / \partial x^{k+n} \partial y^{n_2}$ $(g_k = F^{-1}[q_k])$, the function g_0 should allow of the representation

$$g_0(x,y) = x^{n_*} g_*(x,y), \quad g_*(0,y) \neq 0, \quad g_* \in S(\mathrm{R}^2) \tag{2.11}$$

where $n_* = n - m - 1$. The solution of problem (2.1) in this case is also determined by relations (2.10).

Hence, we have proved the following assertion.

Theorem 2.3. *Suppose*

$$\mathrm{Ind}_\alpha G(\alpha,\beta) = k \geq 0, \quad g = F^{-1}[q] \in S'_n(\mathrm{R}^2)$$

Then, if $m \geq n-1$ $(m = \min\{m_+, m_-\})$, a general solution of problem (2.1) *exists in the subspaces $\Omega'_{\pm,m}(R^2)$ and is determined by relations* (2.10). If $m < n-1$, *then, for the case when solution* (2.10) *exists in $\Omega'_{\pm,m}(R^2)$ it is necessary and sufficient for condition* (2.11) *to be satisfied for $n_* = n - m - 1$.*

Similarly, we can establish the following assertion from Theorems 2.1, 2.2 and 1.2.

Theorem 2.4. *Suppose*

$$\mathrm{Ind}_\alpha G(\alpha,\beta) = k < 0, \quad g = F^{-1}[q] \in S'_n(\mathrm{R}^2)$$

Then, if $m \geq n-1-k$ $(m = \min\{m_+, m_-\})$, a general solution of problem (2.1) *exists in the subspaces $\Omega'_{\pm,m}(R^2)$ and is defined by relations* (2.10). *If $m < n-k-1$, then, for solutions to exist in $\Omega'_{\pm,m}(R^2)$ it is necessary and sufficient for condition* (2.11) *to be satisfied when $n_* = n-k-m-1$.*

Corollary 2.1. *Suppose $m = n-1$. Then, if $k \geq 0$, problem* (1.1) *is solvable in the subspaces $\Omega'_{\pm,m}(R^2)$, while if $k < 0$, it is solvable in specified subspaces when condition* (2.11) *is satisfied when $n_* = -k$. The general solution of problem* (2.1) *is defined by relations* (2.10) *and depends on $m+k$ $(m+k>0)$ derivatives of the functions from the space $S'(R)$.*

3. Solution of the boundary-value problem and construction of the integral relations

The results obtained above enable us to solve matrix problem (1.4). Considering the first three relations of (1.4) as jump problems in $S'(R^2)$ and using Theorems 1.2 and 2.1, we obtain

$$V_k^\pm = i\alpha^{k-2}\beta^{1-k}(i\alpha^{k-1}\beta^{2-k}V_3^\pm + \chi_{0,k}^\pm), \quad k = 1, 2$$
$$V_5^\pm = i\beta^{-1}(i\alpha V_4^\pm + \chi_{0.3}^\pm), \quad \chi_{0,k}^\pm(\beta) = (\chi_{*,k}^+ \pm \chi_{*,k}^-)/2 \tag{3.1}$$

Equation (1.2) then takes the form

$$\begin{Vmatrix} P_4^+ & -\alpha P_3^+ \\ P_3^+ & -\alpha P_2^+ \end{Vmatrix} \begin{Vmatrix} V_3^+ \\ V_4^+ \end{Vmatrix} = -\begin{Vmatrix} P_4^- & -\alpha P_3^- \\ P_3^- & -\alpha P_2^- \end{Vmatrix} \begin{Vmatrix} V_3^- \\ V_4^- \end{Vmatrix} + \begin{Vmatrix} G_1 \\ G_2 \end{Vmatrix} \tag{3.2}$$

where

$$G_k = G_k^+ + G_k^-, \quad G_k^\pm = i\sum_{n=1}^{6} l_{kn}^\pm \chi_{0,k}^\pm(\beta)$$

$$P_2^\pm = \beta_{44}^\pm\beta^2 - 2\beta_{45}^\pm\alpha\beta + \beta_{55}^\pm\alpha^2$$

$$P_3^\pm = \beta_{14}^\pm\beta^3 - (\beta_{15}^\pm + \beta_{34}^\pm)\alpha\beta^2 + (\beta_{24}^\pm + \beta_{35}^\pm)\alpha^2\beta - \beta_{25}^\pm\alpha^3$$

$$P_4^\pm = \beta_{11}^\pm\beta^4 - 2\beta_{13}^\pm\alpha\beta^3 + (\beta_{33}^\pm + 2\beta_{12}^\pm)\alpha^2\beta^2 - 2\beta_{23}^\pm\alpha^3\beta + \beta_{22}^\pm\alpha^4$$

$$l_{11}^\pm = \beta l_1^\pm, \quad l_{12}^\pm = \alpha l_2^\pm, \quad l_{13}^\pm = \alpha l_5^\pm, \quad l_{14}^\pm = -\alpha\beta^2, \quad l_{15}^\pm = \alpha^2\beta, \quad l_{16}^\pm = 0$$

$$l_{21}^\pm = \beta g_1^\pm, \quad l_{22}^\pm = \alpha g_2^\pm, \quad l_{23}^\pm = \alpha g_5^\pm, \quad l_{24}^\pm = 0, \quad l_{25}^\pm = 0, \quad l_{26}^\pm = \alpha\beta$$

$$g_k^\pm = \beta_{4k}^\pm\beta - \beta_{5k}^\pm\alpha, \quad l_k^\pm = \beta_{1k}^\pm\beta^2 - \beta_{3k}^\pm\alpha\beta + \beta_{2k}^\pm\alpha^2; \quad k = 1, 2, \ldots, 5$$

Introducing the functions

$$\Phi_k^\pm(\alpha) = \pm(P_{5-k}^\pm V_3^\pm(\alpha,\beta) - \alpha P_{4-k}^\pm V_4^\pm(\alpha,\beta) - G_k^\pm)$$

we can write matrix equation (3.2) as follows:

$$\Phi_k^+(\alpha) = \Phi_k^-(\alpha), \quad k = 1, 2 \tag{3.3}$$

Since $V_k^\pm \in \Omega'_{\pm,0}(R^2)$, the functions $\Phi_k^\pm(z)(Re z = \alpha)$ are analytic in the upper half-plane (Imz > 0) and in the lower half-plane (Imz < 0) respectively. Moreover, using Theorem 1.2, we conclude that $\lim \Phi_k^\pm(z) = 0$ when $z \to \infty (k = 1, 2)$. These properties and Theorem 2.2 enable us to write $\Phi_k^\pm(\alpha) = 0 (k = 1, 2)$. Hence, taking representations (3.1) into account, we obtain the following expressions for the tranformant of the required functions

$$V_k = V_k^+ + V_k^- \ (V_k = (-i\beta)V_k,\ k = 6, 7, 8); \ V_k^\pm = i(P_5^\pm)^{-1}\sum_{j=1}^{6} r_{kj}^\pm \chi_{0,j}^\pm,\ k = 1, 2, \ldots, 8 \ (3.4) \tag{3.4}$$

where

$$P_5^\pm = (P_3^\pm)^2 - P_2^\pm P_4^\pm, \quad r_{kj}^\pm = h_k^\pm \lambda_{jl}^\pm, \quad k = 1, 2, \ldots, 5, \quad j = 1, 2, \ldots, 6$$

$$r_{6,j}^\pm = \beta\alpha^{-1}(\lambda_{j,1}^\pm l_1^\pm - \lambda_{j,2}^\pm g_1^\pm), \quad r_{7,j}^\pm = \lambda_{j,1}^\pm l_2^\pm - \lambda_{j,2}^\pm g_2^\pm, \quad r_{8,j}^\pm = \lambda_{j,1}^\pm l_5^\pm - \lambda_{j,2}^\pm g_5^\pm$$

$$\lambda_{1,l}^\pm = \alpha^{-1}(g_1^\pm P_{2+l}^\pm - l_1^\pm P_{1+l}^\pm), \quad \lambda_{2,l}^\pm = \beta^{-1}(g_2^\pm P_{2+l}^\pm - l_2^\pm P_{1+l}^\pm),$$
$$\lambda_{3,l}^\pm = \beta^{-1}(g_5^\pm P_{2+l}^\pm - l_5^\pm P_{1+l}^\pm)$$

$$\lambda_{4,l}^\pm = \beta P_{1+l}^\pm, \quad \lambda_{5,l}^\pm = -\alpha P_{1+l}^\pm, \quad \lambda_{6,l}^\pm = P_{2+l}^\pm, \quad \{h_k\}^5 = \{\beta^2, \alpha^2, -\alpha\beta, -\beta, \alpha\},$$

$$l = \begin{cases} 1, & \text{if} \quad k = 1, 2, 3 \\ 2, & \text{if} \quad k = 4, 5 \end{cases}$$

Inversion of the transformant (3.4) leads to expressions for the components of the vector v

$$\upsilon_k = \partial\upsilon_k/\partial y, \quad k = 6, 7, 8$$

$$\upsilon_k = -\frac{1}{\pi}\sum_{p=1}^{6}\left\{\int_{-\infty}^{\infty}\left(\zeta_p^+(t)\mathrm{Im}\sum_{n=1}^{3}\frac{\theta(x)R_{k,p,n}^+}{z_n^+x+y-t}+\zeta_p^-(t)\mathrm{Im}\sum_{n=1}^{3}\frac{\theta(-x)R_{k,p,n}^-}{z_n^-x+y-t}\right)dt\right\}, \quad k = 1, 2, \dots, 8$$

$$R_{k,p,n}^{\pm} = \frac{r_{k,p}^{\pm}(z_n^{\pm}, 1)}{\beta_0^{\pm}q_n^{\pm}(z_n^{\pm})\bar{q}_n^{\pm}(z_n^{\pm})}, \quad \beta_0^{\pm} = \beta_{22}^{\pm}\beta_{55}^{\pm}-(\beta_{25}^{\pm})^2$$

$$q_n^{\pm}(z_n^{\pm}) = \prod_{l=1, l\neq n}^{3}(z_n^{\pm}-z_l^{\pm}), \quad \bar{q}_n^{\pm}(z_n^{\pm}) = \prod_{l=1}^{3}(z_n^{\pm}-\bar{z}_l^{\pm}), \quad P_6^{\pm}(z_n^{\pm}, 1) \equiv 0 \tag{3.5}$$

where $\theta(x)$ is the Heaviside function. Solution (3.5) enables us to establish a relation between the sums and the jumps (1.3) in the plane in which the half-spaces are joined:

$$\boldsymbol{\chi}^+(y) = \mathbf{C}\boldsymbol{\chi}^-(y)+\mathbf{S}\boldsymbol{\Gamma}_R[\boldsymbol{\chi}^-] \tag{3.6}$$

Here

$$\Gamma_R[\chi] \equiv \frac{1}{\pi}\int_R\frac{\chi(t)dt}{t-y}, \quad \mathbf{C} = \{c_{kj}\}^6 = -\mathrm{Re}\{(\mathbf{A}_+ + i\mathbf{B}_-)^{-1}(\mathbf{A}_- + i\mathbf{B}_+)\}$$

$$\mathbf{S} = \{s_{kj}\}^2 = \mathrm{Im}\{(\mathbf{A}_+ + i\mathbf{B}_-)^{-1}(\mathbf{A}_- + i\mathbf{B}_+)\}$$

$$\mathbf{A}_{\pm} = \{a_{kj}^{\pm}\}^6 = \mathrm{Re}(\mathbf{N}^+ \pm \mathbf{N}^-)/2+\mathbf{E}_6, \quad \mathbf{B}_{\pm} = \{b_{kj}^{\pm}\}^6 = \mathrm{Im}(\mathbf{N}^+ \pm \mathbf{N}^-)/2$$

$$\mathbf{N}^{\pm} = \{N_{k,j}^{\pm}\}^6, \quad N_{1,j}^{\pm} = N_{1,j}^{\pm,*}, \quad N_{2,j}^{\pm} = N_{3,j}^{\pm,*}, \quad N_{3,j}^{\pm} = N_{4,j}^{\pm,*}$$

$$N_{k,j}^{\pm} = N_{k+2,j}^{\pm,*}, \quad k = 3, 4, \dots, 6, \quad N_{k,j}^{\pm,*} = \sum_{n=1}^{3}R_{k,j,n}^{\pm}$$

where $\mathbf{E}_6$ is a sixth-order identity matrix.

Equation (3.6) generalizes the relations for a composite anisotropic plane[2] and enables us to reduce different problems for a composite anisotropic space, which is under conditions of generalized plane deformation and weakened by tunnel defects (cracks or inclusions) in the plane in which the materials are bonded, to systems of singular integral equations.

4. Solution of the crack problem

Conditions (1.1) and the equalities

$$\chi_k^-(y) = 0, \quad y\notin l_0; \quad l_0 = \bigcup_{j=0}^{r}l_j, \quad k = 1, 2, \dots, 6$$

which reflect the fact that the half-spaces are joined outside the cracks, lead, using the first three relations of (3.6), to a system of singular integral equations in the derivatives of the jumps of displacements on the cracks

$$\mathbf{C}_*\boldsymbol{\eta}(y)+\mathbf{S}_*\Gamma_{l_0}[\boldsymbol{\eta}] = \mathbf{q}(y), \quad y\in l_0$$

$$\boldsymbol{\eta} = \{\eta_k\}^3 = \{\chi_k^-(y)\}_{k=6}^{8}, \quad \mathbf{q} = \boldsymbol{\chi}_*^+ - \mathbf{C}_0\boldsymbol{\chi}_*^- - \mathbf{S}_0\Gamma_{l_0}[\boldsymbol{\chi}_*^-] = \{q_k\}^3, \quad \boldsymbol{\chi}_*^{\pm} = \{\chi_k^{\pm}(y)\}^3$$

$$\mathbf{C}_* = \{c_{kj}\}_{k=1,2,3,\ j=4,5,6} \quad \mathbf{S}_* = \{s_{kj}\}_{k=1,2,3,\ j=4,5,6}, \quad \mathbf{C}_0 = \{c_{kj}\}_{k=1,2,3,\ j=1,2,3}$$

$$\mathbf{S}_0 = \{s_{kj}\}_{k=1,2,3,\ j=1,2,3} \tag{4.1}$$

System (4.1) must be supplemented by the conditions of crack closure

$$\int_{l_j}\eta_k(t)dt = 0, \quad j = 1, 2, \dots, r, \quad k = 1, 2, 3 \tag{4.2}$$

Inverting[2] system (4.1), we obtain the following expressions for the required functions

$$\boldsymbol{\eta} = \mathbf{T}t, \quad \mathbf{t} = \{t_j\}^3$$

$$t_j(y) = \frac{1}{\omega_j(y)}\{\gamma_j^0[\lambda_j\omega_j(y)g_j(y) - \Gamma_{l_0}[\omega_j g_j]] + \vartheta_j^r(y)\}$$

$$\omega_j(y) = \prod_{k=1}^{r}(b_k - y)^{\mu_j}(y - a_k)^{1-\mu_j}, \quad \vartheta_j^r(y) = \sum_{k=1}^{r-1} c_{jk}^* y^k$$

$$\gamma_j^0 = \frac{1}{\mu_j^2 + 1}, \quad \mu_j = \frac{1}{2\pi i}\ln\frac{\lambda_j + 1}{\lambda_j - 1}, \quad \mathbf{g} = \{g_j\}^3 = \mathbf{Hq}(y), \quad \mathbf{H} = \{h_{k,j}\}^3 = \mathbf{T}^{-1}\mathbf{S}_*^{-1} \tag{4.3}$$

$\mathbf{T} = \{t_{k,j}\}^3$ and λ_j ($j = 1, 2, 3$) are the converting matrix and the eigenvalues of the matrix $\mathbf{S}_*^{-1}C_*$ respectively. The constants c_{jk}^* are found from conditions (4.2). The indicators of a singularity of the solutions at the crack tips have the form

$$\mu_1 = \frac{1}{2}, \quad \mu_{2,3} = \frac{1}{2} \pm i\alpha_1; \quad \alpha_1 = \frac{1}{2\pi}\ln\frac{1+\alpha_0}{1-\alpha_0}, \quad \alpha_0 = \operatorname{Im}\lambda_2, \quad 0 < \alpha_0 < 1 \tag{4.4}$$

which agree with known results.[8,9]

Consider a special case of the problem. Suppose the surfaces of one crack ($r = 1$, $l_0 = (-a, a)$) are loaded with a constant symmetrical load $\boldsymbol{\chi}_*^- = 0$, $\boldsymbol{\chi}_*^+ = 2\{Q_j\}^3$. We will write the jumps in the displacements in this case in the form

$$u_j(y) = \sqrt{a^2 - y^2}\sum_{k=1}^{3} Q_k\left[\varepsilon_{jk} + \rho_{jk}\cos\left(\alpha_1 \ln\left|\frac{a+y}{a-y}\right| - \phi_{jk}\right)\right] \tag{4.5}$$

where

$$u_j'(y) = \eta_j(y), \quad |y| < a$$

$$\xi_{jk} = -2h_{1k}t_{j1}, \quad \rho_{jk} = \sqrt{\varphi_{jk}^2 + \theta_{jk}^2}, \quad \phi_{jk} = \arccos\frac{\varphi_{jk}}{\rho_{jk}}, \quad \varphi_{jk} = -4\frac{\operatorname{Re}(t_{j2}h_{2k})}{\sqrt{1-\alpha_0^2}},$$

$$\theta_{jk} = 4\frac{\operatorname{Im}(t_{j2}h_{2k})}{\sqrt{1-\alpha_0^2}}$$

An analysis of the behaviour of solutions (4.5) at the crack tips for normal loading ($Q_2 = Q_3 = 0$) (in this case the solution obtained will correspond to the problem of the stretching of a half-space at infinity by a symmetrical normal load) shows that, when the condition

$$|\xi_{11}/\rho_{11}| > 1 \tag{4.6}$$

is satisfied, there is no superposition of the crack surfaces, and when it is not satisfied the size δ_0 of the zone where the surfaces overlap can be found from the formula

$$\delta_0/(2a) = (1 + \exp(\alpha_2/|\alpha_1|))^{-1}, \quad \alpha_2 = \phi_{11} + \arccos(-\xi_{11}/\rho_{11}) \tag{4.7}$$

It follows from relations (4.6) and (4.7) that either $\delta_0 = 0$, for example, for many combinations of materials of the monoclinic system, or the value of δ_0 does not go beyond the limit indicated[14] for isotropic materials. Moreover, for known combinations of anisotropic materials[15] the value of δ_0 turns out to be considerably less than this limit value. Hence, in the most general case of a composite anisotropic space, in conditions of generalized plane deformation, the analysis of the behaviour of the solutions in the neighbourhood of the crack tips within the framework of the linear model is valid.

The stresses along the line where the materials outside the crack are connected can be written in the form

$$\upsilon_j(0, y) = \sum_{k=1}^{3} Q_k\left\{\gamma_{jk}^* + \frac{\operatorname{sign} y}{\sqrt{y^2 - a^2}}\left[\operatorname{Re}\left(\gamma_{2jk}(y - 2i\alpha_1)\left|\frac{a+y}{a-y}\right|^{i\alpha_1} + \frac{y\gamma_{1jk}}{2}\right)\right]\right\}, \quad |y| > a$$

$$\gamma_{ljk} = 2h_{lk}\sum_{n=1}^{3}\kappa_{j,3+n}t_{nl}, \quad l = 1, 2; \quad \gamma_{jk}^* = \sum_{m,n=1}^{3}\kappa_{j,3+n}t_{nm}h_{mk}, \quad j = 1, 2, 3$$

$$2\{\kappa_{j,n}\}^6 = \operatorname{Re}\mathbf{N}^+(\mathbf{C} + \mathbf{E}_6) + \operatorname{Im}\mathbf{N}^+\mathbf{S} \tag{4.8}$$

If $y \to a+0$, the stresses (4.8) can be represented in the form

$$\upsilon_j(0, y) = \frac{K_j(a)\upsilon_j^a(y)}{\sqrt{y-a}} + \upsilon_j^*(y), \quad 0 < \left|\upsilon_j^a(y)\right| \le 1, \quad \left|\upsilon_j^*(a)\right| < \infty, \quad j = 1, 2, 3 \tag{4.9}$$

where

$$K_j(a) = \sqrt{\frac{a}{2}\sum_{n=1}^{3} k_{jn}^2}, \quad k_{jn} = \sum_{l=1}^{3} Q_l k_{jl}^{*,n}$$

$$k_{jl}^{*,1} = -\frac{1}{2}\gamma_{1,jl}$$

$$k_{jl}^{*,2} = -(\omega_{jl}^*\cos(\alpha_1\ln(2a)) - \omega_{jl}^\circ\sin(\alpha_1\ln(2a)))$$

$$k_{jl}^{*,3} = -(\omega_{jl}^*\sin(\alpha_1\ln(2a)) - \omega_{jl}^\circ\cos(\alpha_1\ln(2a)))$$

$$\omega_{jl}^* = \varsigma_{1jl} + 2\alpha_1\varsigma_{2jl}, \quad \omega_{jl}^\circ = \varsigma_{2jl} - 2\alpha_1\varsigma_{1jl}, \quad \varsigma_{1jl} = \mathrm{Re}\gamma_{2jl}, \quad \varsigma_{2jl} = \mathrm{Im}\gamma_{2jl}$$

When $Q_2 = Q_3 = 0$ the coefficients in formulae (4.9) take the form

$$K_j(a) = Q_1\sqrt{\frac{a}{2}\sum_{n=1}^{3} (k_{j1}^{*,n})^2} \tag{4.10}$$

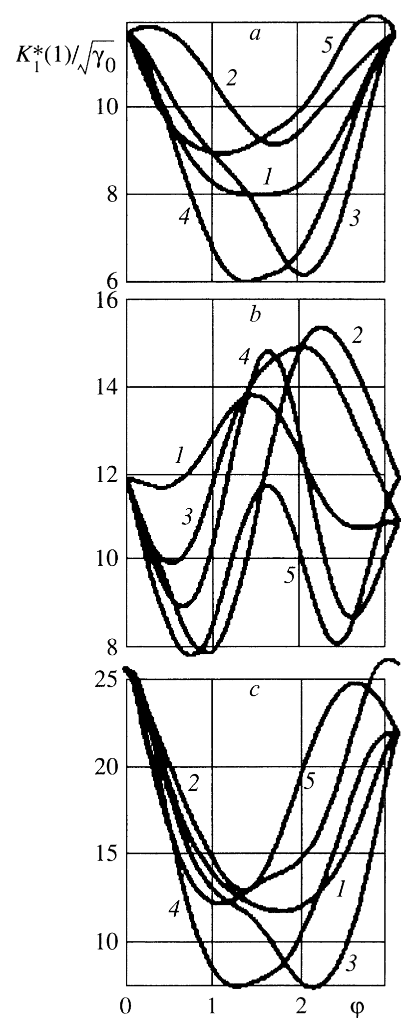

Fig. 1.

The factors $K_j(a)$ depend on the loads, the crack length, and the elastic properties of the half-space, and, consequently, they can be assumed to be a generalization of the stress intensity factors and analogues of the factors considered, for example,[14,16,17] for a composite isotropic plane. If we assume that the crack propagates in the plane in which the half-spaces are joined, then, using the energy approach[16,18] and formula (4.5), we can obtain expressions for the normal fracture stress

$$Q_0 = \frac{\sqrt{8\gamma_0}}{\sqrt{\pi a[\xi_{11} + \sqrt{1-\alpha_0^2}\varphi_{11}(1+4\alpha_1^2)]}} \tag{4.11}$$

where γ_0 is the specific surface energy. Substituting into Eq. (4.10) the value Q_0 instead of Q_1 we obtain the limit value of the factors

$$K_j^*(a) = \left[\frac{4\gamma_0}{\pi(\xi_{11} + \varphi_{11}\sqrt{1-\alpha_0^2(1+4\alpha_1^2)})}\sum_{n=1}^{3}(k_{j1}^{*,n})^2\right]^{1/2}, \quad j = 1, 2, 3 \tag{4.12}$$

on reaching which the crack begins to grow.

5. Results of calculations and their analysis

The behaviour of the relative fracture factor $K_1^*(1)/\sqrt{\gamma_0}$ was investigated for different orthogonal transformations of the principal axes of symmetry[19] at an angle φ of the material of the half-space $x > 0$. The calculations were carried out for a combination of anisotropic materials[15] of the monoclinic system (13 non-zero elastic constants): ethylenediamine tartrate (EDT) (material A), sodium thiosulphate (material B), and a material of the orthorhombic system (9 non-zero elastic constants) topaz (material C).

In the Fig. 1 we show values of the factor $K_1^*(1)/\sqrt{\gamma_0}$ for a space made up of a half-space $x < 0$ of material A and a half-space $x > 0$ of material A (combination *a*), B (combination *b*) or C (combination *c*) with principal axes of anisotropy orthogonally converted by an angle $\varphi(0 \le \varphi \le \pi)$. In all cases all the principal axes of anisotropy of the materials of the half-space $x < 0$ were rotated by an angle $\pi/3$. For half-space $x > 0$ rotation by an angle φ was around the *x* axis (curve 1), the y axis (curve 2), the *x* and *y* axes simultaneously (curve 3), the *y* and I axes simultaneously (curve 4), and around all the axes simultaneously (curve 5).

The results of the calculations show that conversion of the principal axes leads to an increase in the number of non-zero elastic constants (for materials of the monoclinic system their number reaches 21) and has a considerable effect on the value of the relative fracture factor. For combination of materials *a* for all versions of the conversion the minimum value of this factor is reached closer to the middle of the interval, while for the fourth conversion (curve 4) it turns out to be half the maximum value. For combination of materials *b* the greatest ratio of the maximum value of the relative fracture factor to its minimum value is reached for the second conversion (curve 2) and is greater than two. This ratio reaches its greatest value (greater than three) for a combination of materials *c* for the third (curve 3) and fourth (curve 4) conversions of the principal axes of symmetry. Note that for all the conversions, for a combination of materials *b* no contact of the crack surfaces is observed: $\delta_0 = 0$, and for a combination of materials *a* and *c* the value of δ_0 is practically equal to zero (it does not exceed 10^{-20}).

Hence, taking into account the antiplane component has a considerable effect on the behaviour of the relative fracture factor $K_1^*(1)/\sqrt{\gamma_0}$ and on the value of the region of overlap of the surfaces δ_0, which, for certain combinations of the materials, may not, in general, occur.

The proposed method enables one to obtain a solution of the problems in closed form for other types of defects also, for example, peeling and unpeeling inclusions.

References

1. Popov GYa. *The Concentration of Elastic Stresses near Punches, Cuts, Thin Inclusions and Supports*. Moscow: Nauka; 1982.
2. Krivoi AF, Radiollo MV. Singularities of the stress field near inclusions in a composite anisotropic plane. *Izv Akad Nauk SSSR MTT* 1984;**3**:84–92.
3. Wu KC. Stress intensity factor and energy release rate for interfacial cracks between dissimilar anisotropic materials. *Trans ASME Ser EJ Appl Mech* 1990;**57**(4):882–6.
4. Poonsawat P, Wijeyewickrema AC, Karasudni P. Stress singularity analysis of a crack terminating at the interface of an anisotropic layered composite. *Trans ASME Ser EJ Appl Mech* 1998;**65**(4):829–36.
5. Herrmann KP, Lobodda VV. On interface crack models with contact zones situated in an anisotropic biomaterial. *Arch Appl Mech* 1999;**69**(5):317–35.
6. Krivoi AF, Arkhipenko KM. A crack at the interface of two dissimilar anisotropic half-planes materials. *Mat Metody ta Fiz-Mekh Polya* 2005;**48**(3):110–6.
7. Krivoi AF. Arbitrarily oriented defects in a composite anisotropic plane. *Visnik Odes'k Derzh Univ Fiz-Mat Nauki* 2001;**6**(3):108–15.
8. Nazarov SA. The interfacial crack between two bonded anisotropic bodies. Stress singularities and invariant integrals. *Prikl Mat Mekh* 1998;**62**(3):489–502.
9. Nazarov SA. The interfacial crack between two bonded anisotropic bodies. Singularities of the elastic fields and a criterion for fracture when the crack surfaces are in contact. *Prikl Mat Mekh* 2005;**69**(3):520–32.
10. Lekhnitskii SG. *Theory of Elasticity of an Anisotropic Body*. Moscow: Mir; 1981.
11. Brychkov YuA. Smoothness with respect part of to a variables of the solutions of partial differential equations. *Diff Uravneniya* 1974;**10**(2):281–9.
12. Krivoi AF. Fundamental solution for a four-component anisotropic plane. *Visnik Odes'k Derzh Univ Fiz-Mat Nauki* 2003;**8**(2):140–9.
13. Gakhov HD. *Boundary-Value Problems*. New York: Dover; 1990.
14. Erdogan F. Stress Distribution in bonded dissimilar materials with cracks. *Trans ASME Ser EJ Appl Mech* 1965;**32**(2):403–10.
15. Aleksandrov KS, Ryzhova TV. The elastic properties of crystals. A review. *Kristallografiya* 1961;**6**(2):288–314.
16. Salganik RL. Brittle fracture of bonded solids. *Prikl Mat Mekh* 1963;**27**(5):957–62.
17. Rice JR. Elastic Fracture mechanics concepts for interfacial crack. *Trans ASME Ser EJ Appl Mech* 1988;**55**(1):98–103.
18. Cherepanov GP. *Mechanics of Brittle Fracture*. New York: McGraw-Hill; 1979.
19. Chernykh KF. *An Introduction to Modern Anisotropic Elasticity*. New York: Begell House; 1998.

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